

The vacuum vessel used in the experiments had a capacity of about $1\frac{1}{2}$ litres of liquid but was filled with an absorbent packing of asbestos wool to ensure non-spilling when carried. The capacity with absorbent filling was slightly over 1 litre of liquid or 850 litres of gas at N.T.P. The oxygen is delivered through a few turns of copper tube so as to warm up the gas before delivery to the breathing bag.

A simple flow meter is inserted in the circuit (see fig. 3) which indicates at any instant the quantity of gas being supplied. This instrument consists merely of a small vane pivoted so as to swing against the restoring force of a hair spring. The gas enters by way of a wide bore jet and the momentum of the stream causes a deflection of the vane.

The apparatus described was developed in the course of an investigation carried out with the Tyndall Mining Research Studentship of the Royal Society, and the author desires to record his indebtedness to the Tyndall Research Committee for their kind encouragement, also to Sir Joseph Petavel for the use of some of the facilities of the National Physical Laboratory.

*A Simple Apparatus for Approximate Harmonic Analysis and
for Periodicity Measurements.*

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The apparatus described in this paper was devised in order to test an extensive series of engineering data for the existence therein of a suspected periodicity. In the first instance, we applied the periodogram method* but found that the time needed for adequate examination by this method was prohibitive. Several other devices were tried before we arrived, as a development from a suggestion made by Mr. G. Udny Yule, F.R.S., at the present apparatus, which was found to be not only satisfactory as a method for detecting periodicity but was also useful as an harmonic analyser. The apparatus has the advantage of being simple to construct, and inexpensive. Further, the construction is such that there can be no change of zero during its use, and while the reduction of the observed results to absolute measure

* 'Edinburgh Mathematical Tracts,' No. 4 (London, 1915), G. A. Carse and G. Shearer, "A Course in Fourier's Analysis and Periodogram Analysis."

requires the employment of corrections, these corrections are mainly based on simple geometry.

In its simplest form the process of periodogram analysis consists in averaging some considerable number, say $2n$, of ordinates taken at a constant interval h along the curve to be analysed, and repeating this averaging for successive series of ordinates starting at $x = 0$, $x = h$, $x = 2h$, $x = 3h$, and so on. If h is exactly equal to any period existing in the curve (or to a multiple of it) and there is no other period, the average ordinate $\bar{y}$ so obtained will show a periodicity of the same amplitude. As h departs from the period, to a greater or less extent, the amplitude of the curve given by $\bar{y}$ falls off. The existence of a number of periods in the curve to be analysed does not alter the general character of the result; in any case $\bar{y}$ shows a maximum amplitude when h is equal to a period. The problem of periodogram analysis, from the mechanical standpoint, thus reduces to the problem of adding or averaging a number of ordinates on a given curve.

Starting with the idea of effecting this addition from a template cut out to represent the required curve, Mr. Yule suggested the employment of a series of coupled springs to effect the addition. The employment of such springs, however, introduces serious constructional difficulties in making them adjustable for various trial periods, from which difficulties the present arrangement escapes. Owing to the appearance of the apparatus, it is commonly described in our laboratory as "The Harp."

Description of Apparatus. A. Essential Parts.

The general appearance of the Harp is shown in plan, and, as a sectional elevation, in fig. 1. A board of triangular shape, 10 feet in length, on a three-point support, is placed with its wider end just overhanging the edge of the table on which it rests. At this wide end of the board are two groups of pulleys (*pp*), each group being spaced at intervals of 25 mm. on a common axle by means of collars with set screws and intermediate washers; meccano parts are quite suitable for the purpose. The innermost pulleys of the two groups are separated by an interval of 50 mm., where the two axles abut in a common support. The four ends of the two axles lie on the arc of a circle struck from the small end of the board, and thus, in view of the small angle subtended by these pulleys (9° of arc), they may be considered effectively as rotating each in a plane which passes through the centre of the circle thus struck, while in number there are effectively nineteen, spaced at equal intervals, but with the centre one missing.

Over each of these pulleys passes a strand of sewing cotton, bearing a

weight made of $\frac{1}{2}$ -inch lead piping, and weighing about 80 grm., which hangs down outside the edge of the table. The weight on each one of these

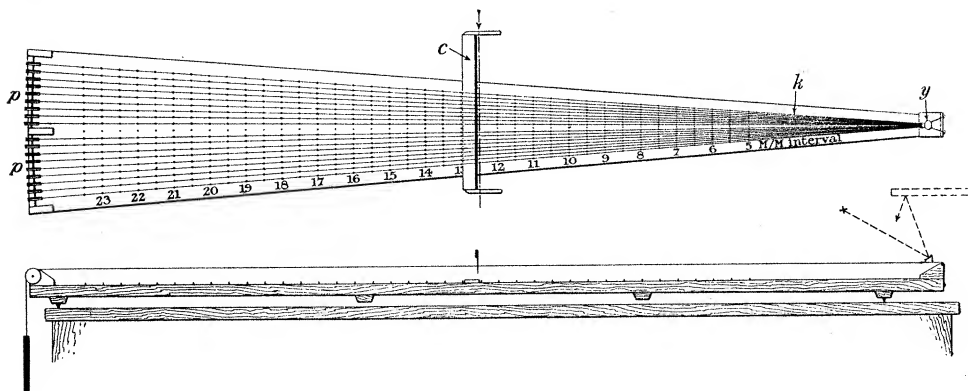


FIG. 1.

strings of the Harp is the same, thus keeping all the strings under the same tension, except for accidental variations in the static friction of the pulleys.

The other ends of these strings converge to the neighbourhood of a common meeting point, each one being inserted into a separate hole arranged for it in a yoke of thin sheet brass at the small end of the board. This yoke (y , figs. 1, 2A, 2B), is drilled with two rows of nine holes each, which alternate. Each string can thus be brought to and threaded through its appropriate hole, then taken back and knotted on itself in a wider part of the board (h), so that, although the eighteen strings are actually in contact laterally, any one may be lifted vertically without disturbing its neighbours.

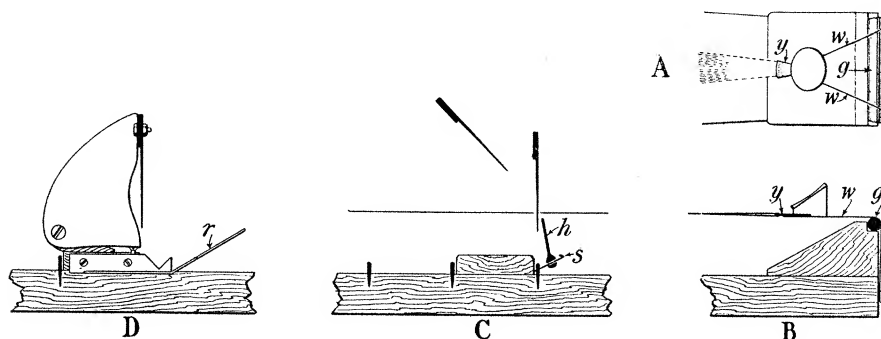


FIG. 2.

The brass yoke (y) is soldered firmly to the apex of a triangular loop of steel piano-wire (w), 0.01 inch thick (figs. 2A, 2B), the ends of which are

fastened by screws to the board after being bent to a right angle round a support of glass rod (g), as shown in the figure. The yoke (y) is thus free to move in a vertical plane under the influence of very small forces, on account of the flexibility of the thin wire, while at the same time it is constrained against horizontal movement or twisting movement. Any such movements of the yoke are magnified and observed by means of a 2-cm. concave mirror, fixed on the yoke in a convenient position by a cork support. A ray of light thrown on the mirror is reflected up to a plane mirror on the ceiling, and thence down to a scale suitably screened by black cloth; this scale may be placed in any convenient position, either by the side of the Harp when the apparatus is being used by a single observer, or elsewhere when two observers are concerned.

Operation of Essential Parts.

The arrangement of strings and yoke thus far described constitutes the essential portion of the apparatus. Considering any one string, this may be regarded as connected to the flexible steel triangle by a hinge at the yoke. Further, the steel triangle is, by virtue of its flexibility, effectively hinged (for movement in a vertical plane) where the steel wire (w) touches the glass rod (g). Under the load of the lead weight hanging from the pulley (p), the yoke hinge lies on a straight line (the weight of the mirror and yoke being negligible) drawn between the upper side of the glass rod and the upper side of the pulley.

Considering all the eighteen strings together, the angle of divergence between the outermost strings is so small that for the purpose of the apparatus they may be considered as effectively parallel to one another at any point along their length. If the angle between the outermost string and the centre line is θ , the outermost string exerts an effective component pull on the yoke of $f \cos \theta$ instead of the pull f exerted by the central string. As θ is only 4° , $\cos \theta$ is 0.995, and the difference of one-half of 1 per cent. is quite negligible. Further, at any such point any number of the strings may be raised above the zero position to any extent required, increase in length of the working portion of the string being permitted by the rise of the weight, the pulley turning to permit this rise, and the tension being unaffected.

If, now, all the eighteen strings are raised to the same extent at some place along the board (*e.g.*, by placing a straight edge underneath the strings), the wire with its yoke and mirror will bend until a new straight line position is assumed, so that the yoke hinge lies in the plane which contains the top of the glass rod and the straight edge respectively. If, on the other hand, only

half the strings are thus raised, and this irrespective of the individual strings chosen, the yoke hinge will rise by only half the previous amount, and the movement of the spot of light reflected from the mirror to the reading scale will also be halved. This strict proportionality will, of course, only hold good so long as the deflection of the mirror is not great enough to be affected by the stiffness of the wire (w), but in the actual operation of the apparatus, this deflection cannot exceed 4° of arc, and is not usually more than 2° .

Similarly, for any number of strings lifted from the neutral position to a given height above it, the position of the mirror will change in proportion to the number of strings lifted.

Again, for any one string lifted, the deflection of the mirror will be proportionate to the amount of the lift, and similarly, again, for any number of strings lifted to varying extents. For small deflections, *i.e.*, those lying within 1° of arc, the deflection of the mirror is practically in direct proportion to the average deflection of the strings above their zero position. For larger deflections of the strings, the movements of the mirror fall off somewhat, but not sufficiently to hamper the use of the appliance.

Having shown the means whereby the spot of light on the observing scale is enabled to indicate the average position of all the eighteen strings, it remains to describe the accessory parts by which the instrument is made convenient for actual use.

Description of Apparatus. B. Accessory Parts.

The triangular board is scaled off to make a series of trial periods by utilising the divergence of the strings. Underneath each string a straight line is ruled, radiating from the common centre below the yoke, and on each of these lines, at regular intervals, a series of steel pegs is driven, as indicated in fig. 1. These pegs thus form a series of rows across the board, beginning with a row whose members are 5 mm. apart, and increasing these intervals by $\frac{1}{2}$ -mm. steps up to $23\frac{1}{2}$ mm. near the pulleys. These pegs serve a double purpose: in the first place they act as sounders in combination with the click arrangement, to be described shortly, while in the second place they serve to register the positions at which the template may be passed under the strings.

The template is passed under the strings by attaching it to a frame which slides on a carriage (*c*, fig. 1). This carriage is not fixed to the board, but can be placed thereon at any one of the positions defined by the rows of steel pegs. In fig. 1, it is represented in position for testing a trial period of $12\frac{1}{2}$ mm., and the line of movement of the template is indicated by an arrow.

This carriage consists of two parts, namely, a comb, and also the guide in which the template slide moves. The comb consists of a large number of steel needles, about 8 cm. in length, obtained by breaking the heads off ordinary drapers' round-headed bonnet pins, which are then set up in a row and soldered on to a strip of $\frac{1}{8}$ -inch brass to form a comb with about eight teeth per centimetre, and of sufficient length to cover the greatest spread of the strings at the wide end of the board. The object of this comb is to prevent any strings from being pushed sideways out of their proper position by friction with the template passing below them. The effect of its use is necessarily to disturb the exact identity of the intervals between the strings, but the error caused thereby is not serious; thus in fig. 3, the comb and the strings are drawn diagrammatically to scale for an interval between the strings of 10 mm. In order to enable this comb to be placed rapidly over the strings at any position along the board, it is mounted on a hinged frame shown in fig. 2D, in such a way that when the comb is lifted up, its points swing clear of the strings; in this position, the whole carriage can be transferred to any other part of the board and placed between two other rows of pegs, whereupon, on lowering the comb, the points of the needles deflect the strings slightly to the right or left if necessary. The friction of

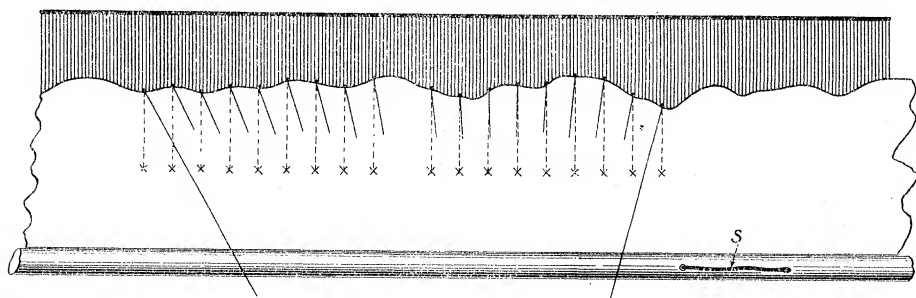


FIG. 3.

the strings on the teeth of the comb introduces a slight error in the nominal equality of tension obtaining between all the strings, but this again is not serious.

The comb also serves to define exactly the position of the edge of the template, since the template leans up against the front of the comb while sliding across the carriage. In fig. 2D, an end elevation of the carriage, is shown one of the V-shaped notches fixed on the end of the wooden base of the carriage, in which the template holder slides. This holder may be of any convenient form, such as a straight bar of metal to which is fixed a wing of tin plate (*h* in fig. 2C), against which the cardboard template can be clipped. The ends of the slider may be equipped with spools on which to

coil up portions of the template not under examination. In addition, this sliding bar bears a clicking spring (figs. 2c and 3, s), so shaped as to make contact with each one in any row of eighteen pins, already described as fastened to the base board, and to emit an audible click on leaving each pin. This slide should be tested in the Harp with a straightedge as a template, to detect any periodicity in the slide.

When the carriage is to be moved to another part of the board for examination of a different trial period, the slide, with its attached template, is turned over (past its dead centre on the strings) until it rests on two projecting arms (fig. 2, r) so as not to foul the under side of the strings while the carriage is being moved.

Operation of Accessory Parts.

A smooth template having been prepared, it is fixed on the slide with its lower edge in contact with the bar thereof. The scale of the template being such that there is reason to expect the occurrence of a period of length lying between 5 and 23 mm., the carriage is then placed on the board, with its base fitting between the 5 and $5\frac{1}{2}$ mm. rows of steel pegs. The slide with its attached template, is then passed under the strings, and placed with its bar resting on the V-notches of the carriage. The template is then rotated in the notches until it makes contact with the strings, and, passing the dead centre, is stopped by contact with the comb as well, where the strings are 5 mm. apart. The slide is then pushed along slowly by hand, thereby causing the strings to rise and fall as they follow the outline of the curve. At every fifth millimetre of this traverse the click is heard. Meanwhile, the operator or another observer is watching the movement of the spot of light. If a period 5 mm. in length actually exists in the template as plotted, the spot of light will oscillate in synchronism with the clicks.

If no such synchronism is noticed between the clicks which are heard and the light movements which are seen, the template is turned down on to its rests, the comb swung up and back, the carriage lifted and replaced at the next interval, and the operation repeated to test for the existence of a period at $5\frac{1}{2}$ mm. Similarly, all periods are tested from one end of the board to the other. The full search over this range, from 5 mm. to 23 mm., by $\frac{1}{2}$ -mm. steps, does not require more than a quarter of an hour to complete.

A trial period having thus been found at which the click and the movement of the light are synchronised, the length of the period is at once shown by the position of the template on the board. The amplitude and the form of the periodic curve can be measured at this setting as follows, preferably by two observers co-operating:—

The slide carrying the template has a millimetre scale marked on it, by means of which the position of the slide is advanced by small and equal successive steps, while the position of the spot of light is recorded for each step on millimetre-squared paper. The height of each ordinate, representing the average position of the strings at each step, may either be chosen arbitrarily, and subsequently corrected, or may be chosen in agreement with a scale previously prepared to show the actual deflection of the spot of light for a given mean movement of the strings at this particular spot on the board; in either way, a value for the amplitude of the periodic curve may be obtained, which is usually within 3 per cent. of the true value.

Preparation of Templates.

One or two simple devices which have been found useful are possibly worth mentioning. The paper employed for the making of templates requires to be in long strips not more than 3 inches in width, and the most convenient way of obtaining this is to have three rolls of paper, such as those from a Burrough's adding machine, pasted together to form a flexible cardboard.

This paper is, of course, not squared for plotting, but curves drawn on squared paper can easily be transferred to it by tracing through typewriter carbon paper.

The scale of the curve should be so chosen that in the forward direction there should be no slope exceeding about 45° , since a steeper slope than this will tend to wedge the strings against the comb, and prevent them from rising and falling smoothly.

If the template, after being traced, is merely cut out with scissors and left thus, the rough edge will tend to make the strings vibrate, and cause an inconvenient flicker of the spot of light. This can be obviated by waxing the edge of the template after cutting, preferably with a hard wax, such as that used for making phonograph records, and finally running a hot iron along the edge, to smooth the wax down to the edge of the paper.

Bearing in mind the necessity for avoiding steep slopes in the curves, the scales to which the curves are plotted should be so chosen as to give a total range of about 3 cm. between the highest and lowest portions, for although the instrument as here described can accommodate curves with an amplitude of 5 cm., it is obvious that the advantage from greater movement of the spot of light will tend to be counter-balanced by excessive distortion of the strings. Again, if any preliminary indication exists as to the probable length of the period, the scale should be so chosen as to make the period fall in the neighbourhood of the middle of the board, for the same reason.

In the event of the search requiring to be continued beyond the limits of 5 mm. and 23 mm. on the scale chosen, the curve must, of course, be re-drawn, and a new template cut to a larger or smaller magnification. While the dimensions here given for the instrument are convenient in an ordinary laboratory, there is no reason why they should not be exceeded by increasing the length of the board, so as to give a greater range of search on any one template, since this range at present is such as necessitates the templates being cut at four-fold magnifications and reductions. In this event, it should be borne in mind that the sensitivity of the instrument decreases as the distance of the template from the mirror increases.

Uses of the Apparatus.

We have already described the method of using the apparatus for determining primarily the length but also the amplitude and form of any periodicity existing in a curve. It will be seen that, by suitable choice of templates, the same procedure can be applied to Harmonic Analysis, the single curve being plotted repeatedly on the template until a sufficient length is available to pass through the instrument, the length of each period being made about 40 mm. In this way, the first four or five harmonics can be detected on the same template, and their amplitude measured by the same procedure as before.

Another occasional use for the apparatus is in the smoothing of curves, which can be passed in template form under the strings at any desired setting, and with any number of strings in use, according to the number of points which we require to include in the smoothing, while a second observer plots the readings of the spot of light as the template moves forward.

Summary.

An account is given of a simple and inexpensive means for constructing and using an apparatus by which periodicities can be determined, harmonic analysis effected, and curves smoothed.

The results obtained (both with respect to length and to amplitude) are only approximately correct, but the error involved need not exceed 3 per cent.

The outstanding advantage of the apparatus is the speed with which determinations may be made. Thus, in the case of determinations of periodicity, some fifty trial periods can be examined in less time than is required for the computation of a single trial period under the periodogram arithmetical method.

I have pleasure in acknowledging the assistance of Mr. G. Udny Yule, F.R.S., while the extensive use of the Harp, which enabled its utility to be fairly judged, has been in the hands of my assistants, Miss E. Lanigan and Mr. H. Hancock, in the Experimental Department of the Fine Cotton Spinners' Association at Manchester.

Multenions and Differential Invariants.

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§ 1. *Preliminary Explanations.*—Since multenions appear almost to have been designed by Nature to serve as an algebra dealing with such matters as differential invariants and relativity, I have thought it desirable at the present moment to present a summary of their properties. Towards the end I propose to enter into more detail in applying them to differential invariants, and to point out some of the special properties of multenions when n , the vector complexity of the system, has the particular value 4.

Clifford first, I believe, applied the subject to elliptic space of three dimensions. His bi-quaternion $q + \omega r$, where $\omega^2 = 1$, is the general member of a sub-algebra of multenions when $n = 4$. In the form of multenions suitable to deal with real questions of Relativity Hamilton's bi-quaternion $q + r\sqrt{-1}$ presents itself in a precisely similar manner, and of this I shall have more to say towards the end of the paper. Joly's 'Manual of Quaternions' has its last chapter devoted wholly to multenions. I believe the most elaborate attempt at development is my own paper in 'Proc. Roy. Soc., Edin.,' 1907-8, p. 503.

The notation used below is based as closely as possible on quaternion notation. It is practically identical with Joly's so far as the latter goes. It differs in several respects from the notation of my own former paper. The changes are all in the direction of simplicity, and back towards quaternion rules. The same applies to changes in terminology. These changes, which I believe are all for the better, are in the main due to friendly advice received from Prof. Knott in connection with the former paper.

Law I. Besides the unit scalar 1 there are given n generators, which will generally be regarded as representing n unit mutually orthogonal vectors in a